Homework #2

Please hand in by next Thursday (October 30), either during the lecture or email to <Philipp.Kleinert@physics.ox.ac.uk>

1. Campbell-Baker-Hausdorff formula: Show that
$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]} + (3^{\text{rd}} \text{ order terms}). \tag{1}$$

2. Consider the free particle path integral (with $m = 1$ for simplicity)
$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q(t)\ \exp\left[i \int_{t_i}^{t_f} \frac{1}{2}\dot{q}^2\, dt \right] \tag{2}$$
Write a general path $q(t)$ as the sum of the classical path[1] $q_c(t)$ plus a Fourier series with coefficients a_n, $n \geq 1$.

3. Show that the action for such a general path is
$$S = \frac{1}{2}\frac{(q_f - q_i)^2}{t_f - t_i} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2}\frac{(n\pi)^2}{t_f - t_i} a_n^2 \tag{3}$$

4. Perform the integral
$$\int da_k\ e^{iS} \tag{4}$$
over a single Fourier mode.

5. Write the entire path integral as a constant, depending only on $t_f - t_i$, times the classical action,
$$\int \prod_{n=1}^{\infty} a_n\ e^{iS} = c(t_f - t_i) \exp\left(\frac{i}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right) \tag{5}$$
Does the constant have a finite value?

6. The actual path integral measure contains a normalization constant γ,
$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}q e^{iS} = \gamma \int \prod da_n\ e^{iS}, \tag{6}$$
such that the combination $\gamma \cdot c(t_f - t_i)$ is a finite number. The requirement that
$$\int dq\ \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle = \langle q_f, t_f | q_i, t_i \rangle \tag{7}$$
implies a relation between $\gamma c(t_f - t)$, $\gamma c(t - t_i)$, and $\gamma c(t_f - t_i)$. Find it and solve it. Hint: $\gamma c(\tau) \sim \tau^{-1/2}$.

[1] That is, motion at constant velocity.